RECONSTRUCTION OF FUNCTION FIELDS
NOTES BY VALENTIJN KAREMAKER

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1. Introduction and set-up

Let ℓ be a prime number and let k be an algebraically closed field with $\mathrm{char}(k) \neq \ell$. Let K/k be a function field of transcendence degree $\mathrm{td}(K/k) > 1$ with absolute Galois group G_K.

Take $G_K^{(1)} = G_K$ and for all $i \geq 1$, let $G_K^{(i+1)} = [G^{(i)}, G^{(1)}](G^{(i)})^{\ell^\infty}$. In other words, the groups $\{G_K^{(i)}\}_{i\geq 1}$ form the descending central ℓ^∞-series of G_K.

Then

$$\Pi_K^a = G^{(1)}/G^{(2)} = \mathrm{Gal}(K'/K)$$

is the Galois group of the maximal pro-ℓ abelian extension K' of K, and

$$\Pi_K^c = G_K^{(1)}/G_K^{(3)} = \mathrm{Gal}(K''/K)$$

is the Galois group of the maximal pro-ℓ abelian-by-central extension K'' of K. Note that the projection map $pr : \Pi_K^c \to \Pi_K^a$ has kernel $[\Pi_K^c, \Pi_K^c]$, and therefore Π_K^a can be group theoretically recovered from Π_K^c.

We will prove the following theorem, following the results of Pop in [1–3].

1.1. Theorem. *[1, Theorem I] Assume that $k = \overline{k}$ is the algebraic closure of a finite field. With the notation as above, K/k can be group theoretically recovered from Π_K^c.*

The proof consists of two parts:

- The **local theory**: starting from Π_K^c, this yields the decomposition groups and inertia groups in Π_K^a corresponding to prime divisors (valuations) of K/k. The output of the local theory is the so-called total decomposition graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ of K/k, together with its rational quotients; cf. Corollary 7.2 and Proposition 7.3.
- The **global theory**: starting from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ and some of its (rational) quotients, this yields the projective space $\mathbb{P}(K) = K^\times/k^\times$ inside $\hat{K} = \mathrm{Hom}_{\mathrm{cont}}(\Pi_K^a, \mathbb{Z}_\ell)$, together with all projective lines. Using Artin's fundamental theorem of projective geometries, one recovers K/k from this; cf. Theorem 6.11.

1.2. Remark. The paper [4] proves a generalisation of Theorem 1.1: if $\mathrm{td}(K/k) > \dim(k)+1$ (where $\dim(k)$ denotes the Kronecker dimension of k), then K/k can be group theoretically recovered from $\Pi_{K/k}^c$. Since $\dim(k) = 0$ if and only if k is the algebraic closure of a finite field, this implies Theorem 1.1.

Let the divisorial inertia $\mathfrak{In}.\mathfrak{div}(K)$ of K be defined as in Definition 2.6. The proof in [4] works as follows: first, [4, Theorem 1.2(2)] group theoretically recovers $\mathfrak{In}.\mathfrak{div}(K)$ from $\Pi^c(K)$ if $\mathrm{td}(K/k) > \dim(k)$. Then [4, Theorem 1.1(1)] recovers K/k from $\Pi^c(K)$ *endowed with* $\mathfrak{In}.\mathfrak{div}(K)$, again by first recovering the total decomposition graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ and then reconstructing the projective space $\mathbb{P}(K)$ and projective lines.

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2. Valuation theory

2.1. Notation. For any valuation v of K/k (that is to say, a valuation of K which is trivial on k), let Kv be its residue field, vK be its value group, and $\mathscr{O}_v$ its valuation ring. Valuations are ordered by $v \leq w \leftrightarrow \mathscr{O}_w \subseteq \mathscr{O}_v$. Then v is a *coarsening* of w, and w is a *refinement* of v by some valuation $v_0 = w/v$ on Kv. In other words, $w = v_0 \circ v$ is the valuation theoretic composition of v and v_0.

2.2. Definition (Prime r-divisors)**.** A *prime r-divisor* or *generalised prime divisor* of K/k is a valuation v of K/k which satisfies $\mathrm{td}(Kv/k) = td(K/k) - r$ and for which there exists a chain of valuations $v_1 < \ldots < v_r := v$. If $r = 1$, then v is a *prime divisor*. A prime r-divisor satisfies $v = v_r \circ \ldots \circ v_1$ where v_1 is a prime divisor of K and v_{i+1} is a prime divisor of Kv_i for all $i \geq 1$.

2.3. Notation. For a valuation v of K/k, denote by $T_v \leq \Pi_K^a$ its (abelian) inertia group and by $Z_v \leq \Pi_K^a$ its (abelian) decomposition group. More precisely, let v'/v be a(ny) prolongation of v from K to K' and let $T_v = T_{v'/v}$ and $Z_v = Z_{v'/v}$. For any prime (r-)divisor v, the group Z_v endowed with T_v is an *(r-)divisorial subgroup* of Π_K^a. And for any v, there is an exact sequence

$$1 \to T_v \to Z_v \to \Pi_{Kv}^a \to 1. \tag{1}$$

We have $v = w$ if and only if $T_v = T_w$ if and only if $Z_v = Z_w$, and $v < w$ if and only if $T_v \subsetneq T_w$ if and only if $Z_v \supsetneq Z_w$. If $v < w$, there are exact sequences

$$1 \to T_v \to T_w \to T_{w/v} \to 1$$

and

$$1 \to T_v \to Z_w \to Z_{w/v} \to 1.$$

For a prime r-divisor v, we have $T_v \simeq \mathbb{Z}_\ell^r$ (or more precisely, $T_v \simeq \mathbb{T}_{\ell,K}^r$ where $\mathbb{T}_{\ell,K}$ is the ℓ-adic Tate module of $K^\times$). If v is a prime r-divisor, w is a prime s-divisor, and $v < w$, then $T_w/T_v \simeq \mathbb{Z}_\ell^{s-r}$.

2.4. Definition (Flags of divisors/divisorial subgroups)**.**

(1) A *flag of prime r-divisors* is a sequence $v_1 \leq \ldots \leq v_r$ such that each v_m is a prime m-divisor of K/k for $1 \leq m \leq r$.
(2) To a flag as above one associates a *flag of r-divisorial subgroups*, which is a sequence $Z_{v_1} \geq \ldots \geq Z_{v_r}$ of the corresponding decomposition groups, endowed with a sequence $T_{v_1} \leq \ldots \leq T_{v_r}$ of the corresponding inertia groups.

2.5. Lemma. *[2, Proposition 4.2, (1) and (2)] Let v be a prime r-divisor of K/k and let $\widetilde{v}$ denote a prolongation to K'', with inertia resp. decomposition group $T_{\widetilde{v}} \subseteq Z_{\widetilde{v}}$ and residue field $K''\widetilde{v}/Kv$. The following hold, cf. Notation 2.3.*

(1) *Then $T_{\widetilde{v}} \simeq \mathbb{Z}_\ell^r$ and $Z_{\widetilde{v}} \simeq T_{\widetilde{v}} \times \mathrm{Gal}(K''\widetilde{v}/Kv)$.*
(2) *We have $T_v \simeq T_{\widetilde{v}}$ and $\mathrm{Gal}(K''\widetilde{v}/Kv) \twoheadrightarrow \Pi_{Kv}^a$.*

2.6. Definition ((Divisorial) inertia elements)**.**

(1) An element $\sigma \in \Pi_K^a$ is an *inertia element* if there is a valuation v of K such that $\sigma \in T_v$. These elements form the closed subset $\mathfrak{In}(K) \subseteq \Pi_K^a$.
(2) Restricting to valuations which correspond to prime divisors yields the *divisorial inertia* $\mathfrak{In}.\mathfrak{div}(K) \subseteq \mathfrak{In}(K)$. Since the divisorial valuations are dense in the space of all valuations [5, Theorem B], $\mathfrak{In}(K)$ is the topological closure of $\mathfrak{In}.\mathfrak{div}(K)$ in Π_K^a.

2.7. Definition (Core of a valuation)**.** Let v be a valuation of K and choose Λ such that $K \subset K^{Z_v} \subset \Lambda \subset K'$. Let $\mathscr{V}_{v,\Lambda}$ be the set of coarsenings w of v such that $\mathrm{Gal}(K'/\Lambda) \subseteq T_w$. Then let $\mathscr{V}^0_{\Lambda,v} := \mathscr{V}_{\Lambda,v} \cup \{v\}$ and define $v_\Lambda := \inf \mathscr{V}^0_{\Lambda,v}$ to be the *abelian pro-ℓ Λ-core* of v; its valuation ring is the union of all valuation rings $\mathscr{O}_w$ with $w \in \mathscr{V}^0_{\Lambda,v}$.

2.8. Proposition (Properties of v_Λ)**.** *[2, Propositions 2.4 and 2.5]*

(1) *If $\Lambda \neq K'$ is a proper subextension, then v_Λ is nontrivial.*
(2) *If $w \in \mathscr{V}_{\Lambda,v}$ then any prolongation w' to K' satisfies $\Lambda w' = (Kw)'$, where the latter is the maximal pro-ℓ of Kw. And if any v_1 satisfies $v_1 < v_\Lambda$ then $\Lambda v_1' \neq (Kv_1)'$.*
(3) *If v is a prime divisor, then $v = v_{K^{Z_v}}$.*
(4) *If v_1, v_2 are valuations such that $K^{Z_{v_1}}, K^{Z_{v_2}} \subset \Lambda \neq K'$ (in particular the valuations are not independent and have a common coarsening) then $v_{1,\Lambda}$ and $v_{2,\Lambda}$ are comparable.*

3. Commuting-liftable elements

3.1. Definition (Commuting-liftable)**.** For $\Sigma = (\sigma_i)_i \subset \Pi^a_K$, let $\Delta_\Sigma \subseteq \Pi^a_K$ be the closed group it generates. For any closed subgroup $G \subset \Pi^a_K$, let G' be its preimage in Π^c_K under $pr : \Pi^c_K \to \Pi^a_K$.

(1) Σ is *commuting-liftable* if Δ'_Σ is abelian. In fact, any subgroup G of Π^a_K is commuting-liftable if G' is abelian.
(2) A family $(\Delta_i)_i$ of closed subgroups of Π^a_K is *commuting liftable* if $[\Delta'_i, \Delta'_j] = \{1\}$ for all $i \neq j$.

3.2. Remark. The following things follow immediately from Definition 3.1

(1) For $T \subseteq Z$ inside Π^a_K, both T and (T, Z) are commuting-liftable if and only if $T' \subseteq Z(Z')$, where $Z(Z')$ denotes the centre of Z'.
(2) Any closed subgroup Z of Π^a_K has a unique maximal closed subgroup T that satisfies the above equivalent conditions, namely $T = pr(Z(Z'))$.
(3) For any prime r-divisor v of K, both T_v and (T_v, Z_v) are commuting-liftable; this follows from Lemma 2.5. In fact, T_v is the unique maximal subgroup of Z_v such that both T_v and (T_v, Z_v) are commuting-liftable, for $T'_v = Z(Z'_v)$.

3.3. Lemma. *[2, Proposition 4.2(3)(b)] Let v be a prime r-divisor of K/k, where $\mathrm{td}(K/k) = d$. Then Z_v contains commuting-liftable subgroups $\simeq \mathbb{Z}^d_\ell$, and Z_v is the maximal closed subgroup of Π^a_K containing commuting-liftable subgroups $T \simeq \mathbb{Z}^r_\ell$ such that T and (T, Z) are both commuting-liftable; cf. Remark 3.2(3).*

The link between commuting-liftable elements and valuations is provided by the following proposition.

3.4. Proposition. *[6, Corollary 6.4.2] If $\sigma, \tau \in \Pi^a_K$ are commuting-liftable and $\langle \sigma, \tau \rangle$ is not pro-cyclic, then there exists a valuation v of K such that $\langle \sigma, \tau \rangle \subseteq Z_v$ and such that $\langle \sigma, \tau \rangle \cap T_v \neq \{1\}$ (and $\mathrm{char}(Kv) \neq \ell$).*

3.5. Proposition. *[2, Fact 3.3 and Proposition 3.4]*

(1) *For every inertia element $1 \neq \sigma \in \Pi^a_K$, let $\Lambda = (K')^\sigma$. Then there is a minimal valuation v_σ with respect to which σ is an inertia element, namely, the abelian pro-ℓ Λ-core of any (i.e. every, by comparability of cores from Proposition 2.8(4)) valuation v such that $\sigma \in T_v$. This is called the* canonical valuation *for σ.*
(2) *For a family $\Sigma = (\sigma_i)_i$ of commuting-liftable inertia elements, one can form the supremum $v_\Sigma = \sup_i v_{\sigma_i}$ of their canonical valuations. Then $\sigma_i \in T_{v_\Sigma}$ for all i.*

(3) *For $Z \subseteq \Pi_K^a$ a closed subgroup, let $\Sigma_Z = (\sigma_i)_i$ be all inertia elements $\sigma_i \in Z$ such that (σ_i, Z) is commuting-liftable for all i. Then $Z \subseteq Z_{v_\Sigma}$ and $\Sigma_Z = Z \cap T_{v_\Sigma}$.*

3.6. Corollary. *[2, Proposition 3.5]*

(1) *Let $\Delta \subseteq \Pi_K^a$ be a commuting-liftable subgroup. Then Δ contains a subgroup Σ of inertia elements such that Δ/Σ is pro-cyclic (potentially trivial). Hence, there exists a valuation v_Σ such that $\Delta \subseteq Z_v$ and $\Delta \cap T_{v_\Sigma} = \Sigma$.*

(2) *For every closed subgroup $Z \subseteq \Pi_K^a$, consider the maximal subgroup $\Sigma_Z \subseteq Z$ such that both Σ_Z and (Σ_Z, Z) are commuting-liftable. If $\Sigma_Z \neq Z$, then Σ_Z is unique with these properties and consists of all inertia elements $\sigma \in Z$ such that (σ, Z) is commuting-liftable. In particular, $Z \subseteq Z_{v_\Sigma}$ and $\Sigma_Z = Z \cap T_{v_\Sigma}$.*

4. Divisor graphs and decomposition graphs

4.1. Definition (Divisor and decomposition graphs)**.**

(1) The vertices of the *total prime divisor graph* $\mathscr{D}_K^{\mathrm{tot}}$ are the residue fields of all prime r-prime divisors of K/k. Its edges are non-oriented self-edges (corresponding to the trivial valuation) and oriented edges $Kv \to Kw$ if $v = v_r \circ \ldots \circ v_1$ and $w = w_s \circ \ldots \circ w_1$ such that $s = r + 1$ and $v_i = w_i$ for all $1 \leq i \leq r$.

(2) A *geometric prime divisor graph* $\mathscr{D}_K \subset \mathscr{D}_K^{\mathrm{tot}}$ is a connected subgraph for which all non-trivial edges originating in a vertex Kv are *all* prime divisors corresponding to a particular quasi-projective normal model of K/k. Any maximal branch has length $\mathrm{td}(K/k)$ and starts at the vertex K.

(3) The vertices of the *total decomposition graph* $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ are the pro-ℓ groups Π_{Kv}^a for all generalised prime divisors of K/k. If there exists an edge from Kv to Kw in $\mathscr{D}_K^{\mathrm{tot}}$, then there exists an edge from Π_{Kv}^a to Π_{Kw}^a, endowed with the data $T_{w/v} \subseteq Z_{w/v}$.

(4) A *geometric decomposition graph* $\mathscr{G}_{\mathscr{D}_K}$ is a subgraph of $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$, corresponding to a geometric prime divisor graph $\mathscr{D}_K \subset \mathscr{D}_K^{\mathrm{tot}}$.

(5) The vertices of a *level-δ pro-ℓ abstract decomposition graph* $\mathscr{G}$ (with origin $G = G_0$) are pro-ℓ abelian groups G_i. Every edge is called a valuation and it is labelled with a pair of pro-ℓ groups $T_v \subseteq Z_v$ (called the inertia resp. decomposition groups). In particular, every vertex G_i has a unique non-oriented self-edge (called the trivial valuation), labelled with $\{1\} = T_{v_i 0}$ and $Z_{v_i 0} = G_i$, and these are the only cycles in the graph. And for $i \neq j$, there exists at most one oriented edge $G_j \to G_i$ (called a non-trivial valuation), labelled with subgroups $T_{v_i} \subseteq Z_{v_i}$ of G_j such that $T_{v_i} \simeq \mathbb{Z}_\ell$ and $Z_{v_i}/T_{v_i} = G_i$; maximal branches of these have length δ. For $G_j \to G_i$ and $G_j \to G_{i'}$ (with $i \neq i'$) we have $Z_{v_i} \cap Z_{v_{i'}} = \{1\} = T_{v_i} \cap T_{v'_i}$. Finally, for each G_j, there exist systems $(\mathfrak{U}_{i,\alpha})_\alpha$ of cofinite subsets of oriented edges originating at G_j, such that every finite subset of such edges is contained in the complement of some $\mathfrak{U}_{i,\alpha}$ (the system is *cofinal*), and such that the closed subgroup $T_{\mathfrak{U}_{i,\alpha}} = \langle T_{v_i} : v_i \in \mathfrak{U}_{i,\alpha} \rangle$ satisfies $T_{v_i} \cap T_{\mathfrak{U}_{i,\alpha}} = \{1\}$ for all i and $v_i \notin \mathfrak{U}_{i,\alpha}$.

(6) For $\mathscr{G}$ as in the previous item, consider the valuations v of $G = G_0$, with inertia groups T_v. Now $\mathscr{G}$ is *complete curve-like* if each T_v has a generator τ_v such that the system $\mathfrak{I} = (\tau_v)_v$ satisfies $\prod_v \tau_v = 1$ as its only pro-relation. (Any two such systems must be fixed ℓ-adic unit powers of each other.)

(7) An *r-residual abstract decomposition graph* $\mathscr{G}_v$ is the unique maximal connected subgraph of $\mathscr{G}$ with origin $G_r = G_v$ for some $v = (v_r, \ldots, v_1)$ a path of length r originating at G_0. It has level $\delta - r$. Note that v is a prime r-divisor.
We say that $\mathscr{G}$ as in (5) is *level-δ complete curve-like* if all its level-$(\delta - 1)$ residual subgraphs are complete curve-like.

4.2. Proposition (Geometric decomposition graph is abstract decomposition graph)**.** *A geometric decomposition graph $\mathscr{G}_{\mathscr{D}_K}$ is a level-δ pro-ℓ abstract decomposition graph for $\delta \leq \mathrm{td}(K/k)$. (This follows immediately from the definitions and standard valuation theory.)*

4.3. Definition (Abstract divisor group and divisorial decomposition graph)**.**

(1) For any valuation v of $G_0 = G$, let T_v be its inertia group, and define $\widehat{\Lambda}_{T_v} = \mathrm{Hom}(T_v, \mathbb{Z}_\ell)$ and $\widehat{\Lambda}_{\mathscr{G}} = \mathrm{Hom}(G_0, \mathbb{Z}_\ell)$. The *$\ell$-adic abstract divisor group of $\mathscr{G}$* is $\widehat{\mathrm{Div}}_{\mathscr{G}} = \widehat{\oplus_v} \widehat{\Lambda}_{T_v}$. It sits inside an exact sequence

$$0 \to \widehat{U}_{\mathscr{G}} \to \widehat{\Lambda}_{\mathscr{G}} \to \widehat{\mathrm{Div}}_{\mathscr{G}} \to \widehat{\mathfrak{Cl}}_{\mathscr{G}} \to 0 \tag{2}$$

where $\widehat{U}_{\mathscr{G}} = \mathrm{Hom}(G_0/\langle T_v : v \text{ valuation of } G\rangle, \mathbb{Z}_\ell)$ is the *unramified part* of $\widehat{\Lambda}_{\mathscr{G}}$ and $\widehat{\mathfrak{Cl}}_{\mathscr{G}}$ is the *ℓ-adic abstract divisor class group* of $\mathscr{G}$.

(2) Let $\mathfrak{I} = (\tau_v)_v$ be a system of inertia generators (for all valuations v of $G_0 = G$) whose only relation is $\prod_v \tau_v = 1$, as before. Let $\mathscr{F}_{\mathfrak{I}}$ be the abelian pro-ℓ free group on $\mathfrak{I}$; then $\mathrm{Hom}(\mathscr{F}_{\mathfrak{I}}, \mathbb{Z}_\ell) \simeq \widehat{Div}_{\mathscr{G}}$. Construct a corresponding system $\mathfrak{B} = (\phi_v)_v$ of $\phi_v \in \mathrm{Hom}(\mathscr{F}_{\mathfrak{I}}, \mathbb{Z}_\ell)$ such that $\phi_v(\tau_w) = 1$ if $v = w$ and $\phi_v(\tau_w) = 0$ otherwise. Now define the lattice $\mathrm{Div}_{\mathfrak{I}} = \langle \mathfrak{B} \rangle_{(\ell)} \subset \widehat{\mathrm{Div}}_{\mathscr{G}}$; it is a free $\mathbb{Z}_{(\ell)}$-module which is ℓ-adically dense in $\widehat{\mathrm{Div}}_{\mathscr{G}}$. In particular it satisfies $\mathrm{Div}_{\mathfrak{I}} \otimes \mathbb{Z}_\ell = \oplus_v \Lambda_{T_v}$. Any two such lattices are ℓ-adically equivalent, i.e. they are equal up to multiplication by an ℓ-adic unit. Its preimage $\Lambda_{\mathfrak{I}}$ (cf. (2)) in $\widehat{\Lambda}_{\mathscr{G}}$ is a $\widehat{U}_{\mathscr{G}}$-lattice.
Such a lattice may not exist. If it does, *and* if $\mathrm{Div}_{\mathscr{G}_v}$ exists for all residual abstract decomposition graphs $\mathscr{G}_v$, and some other technical conditions hold [3, Fact 8 and Definition 9], then $\mathrm{Div}_{\mathscr{G}} := \mathrm{Div}_{\mathfrak{I}}$ is called an *abstract divisor group*, and $\mathscr{G}$ is a *divisorial abstract decomposition graph*. Equation (2) yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{U}_{\mathscr{G}} & \longrightarrow & \Lambda_{\mathscr{G}} & \longrightarrow & \mathrm{Div}_{\mathscr{G}} & \longrightarrow & \mathfrak{Cl}_{\mathscr{G}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{U}_{\mathscr{G}} & \longrightarrow & \widehat{\Lambda}_{\mathscr{G}} & \longrightarrow & \widehat{\mathrm{Div}}_{\mathscr{G}} & \longrightarrow & \widehat{\mathfrak{Cl}}_{\mathscr{G}} & \longrightarrow & 0. \end{array} \tag{3}$$

4.4. Definition (Morphisms of abstract decomposition graphs)**.**
Let $\mathscr{G}$ and $\mathscr{H}$ be two abstract decomposition graphs, of levels $\delta_{\mathscr{G}}$ and $\delta_{\mathscr{H}}$, with origins G_0 and H_0, trivial valuations v_0 and w_0, and 1-residual abstract decomposition subgraphs $\mathscr{G}_v$ and $\mathscr{H}_w$, respectively.
Let $\Phi : G_0 \to H_0$ be a continuous group homomorphism. We say that w_0 corresponds to v if and only if $\Phi(T_v) = 1$ and $\Phi(Z_v)$ is open in H_0.
More generally, $w = (w_s, \ldots, w_1) = (\mathfrak{w}_1, w_1)$ corresponds to $v = (v_r, \ldots, v_1) = (\mathfrak{v}_1, v_1)$ if either $\Phi(T_{v_1}) = 1$ (then in fact w corresponds to $\mathfrak{v}_1$), or $\Phi(T_{v_1}) \neq 1$ but $\Phi(T_{v_1}) \subseteq T_{w_1}$ and $\Phi(Z_{v_1}) \subseteq Z_{w_1}$ are open subgroups (then $\mathfrak{w}_1$ corresponds to $\mathfrak{v}_1$).
For any $\delta \leq \delta_{\mathscr{G}}, \delta_{\mathscr{H}}$, we inductively define a *level-δ morphism* $\Phi : \mathscr{G} \to \mathscr{H}$:

(1) A *level-0 morphism* is a group homomorphism $\Phi : G_0 \to H_0$ such that w_0 corresponds to v_0.

(2) Almost all 1-residual vertices of $\mathscr{H}$ correspond to some (finitely many) 1-residual vertices of $\mathscr{G}$. If w_0 corresponds to a 1-residual edge v, then there is a level-δ morphism $\Phi_v : \mathscr{G}_v \to \mathscr{H}$. And if a 1-residual edge w corresponds to a 1-residual edge v, then there is a level-$(\delta - 1)$-morphism $\Phi_v : \mathscr{G}_v \to \mathscr{H}_w$.

We say Φ is *proper* if each w corresponds to some v and if, for every 1-residual edge v of $\mathscr{G}$ and $w = \Phi(v)$, the residual morphism $\Phi_v : \mathscr{G}_v \to \mathscr{H}_w$ is proper. Then $\mathscr{H}$ is a level-δ quotient of $\mathscr{G}$ via Φ, and for *any* v and w corresponding to each other under Φ the residual morphism is proper.
Finally, we say that Φ is *divisorial* if all residual morphisms $\Phi_v : \mathscr{G}_v \to \mathscr{H}_w$ (with w of length $< \delta$) satisfy $\hat{\phi}(\widehat{\Lambda}_{\mathscr{H}}) \subseteq \widehat{\Lambda}_{\mathscr{G}}$. Note: there may be non-divisorial morphisms between divisorial abstract decomposition graphs.

4.5. Lemma. *Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-δ morphism of abstract decomposition graphs. Then this induces an injective Kummer morphism $\hat{\phi} : \widehat{\Lambda}_{\mathscr{H}} \to \widehat{\Lambda}_{\mathscr{G}}$ and injective residual Kummer morphisms $\hat{\phi}_v : \widehat{\Lambda}_{\mathscr{H}_w} \to \widehat{\Lambda}_{\mathscr{G}_v}$ (for all w such that $\delta_w < \delta$).*
Morever, since $\Phi(T_v) \subseteq T_w$ is open, if we let v and w be 1-residual vertices and choose inertia generators τ_v and τ_w, then $\Phi(\tau_v) = \tau_w^{a_{vw}}$ for a unique $a_{vw} \in \mathbb{Z}_\ell$.
The map $\hat{\phi}$ in turn induces a morphism $\mathrm{div}_\Phi : \widehat{\mathrm{Div}}_{\mathscr{H}} \to \widehat{\mathrm{Div}}_{\mathscr{G}}$.

4.6. Lemma. *[3, Proposition 30] Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-δ morphism of abstract decomposition graphs. If $\Phi_v : \mathscr{G}_v \to \mathscr{H}_w$ is divisorial for all w of length $\delta_{\mathscr{H}} - 1$, then Φ is divisorial. If Φ is an isomorphism, then it is divisorial, and $\hat{\phi}$ is also an isomorphism.*

5. Rational quotients

5.1. Definition (Rational quotients)**.**

(1) For a level-1 complete curve-like abstract decomposition graph $\mathscr{G}_\alpha$ and a system $\mathfrak{I}_\alpha = (\tau_v)_v$ of generators (cf. Definition 4.1(5,6)), the sequence in (2) becomes

$$0 \to \widehat{U}_{\mathscr{G}_\alpha} \to \Lambda_{\mathfrak{I}_\alpha} \to \mathrm{Div}_{\mathfrak{I}_\alpha} \to \mathfrak{Cl}_{\mathfrak{I}_\alpha} \simeq \mathbb{Z}_{(\ell)} \to 0. \tag{4}$$

Then $\mathscr{G}_\alpha$ is *rational* if $\widehat{U}_{\mathscr{G}_\alpha} = 0$.

(2) Starting from a morphism $\Phi_\alpha : \mathscr{G} \to \mathscr{G}_\alpha$, let $\hat{\phi}_\alpha : \widehat{\Lambda}_{\mathscr{G}_\alpha} \to \widehat{\Lambda}_{\mathscr{G}}$ be the induced Kummer morphism and let $\widehat{\Lambda}_\alpha = \hat{\phi}_\alpha(\widehat{\Lambda}_{\mathscr{G}_\alpha}) \subseteq \widehat{\Lambda}_{\mathscr{G}}$. Then Φ_α is a *rational quotient* of $\mathscr{G}$ if $\mathscr{G}_\alpha$ is rational, Φ_α is divisorial, and for all v such that $\widehat{\Lambda}_\alpha \subset \widehat{U}_v$, then $\widehat{\Lambda}_\alpha$ gets mapped injectively to $\widehat{\Lambda}_{\mathscr{G}_v}$.

(3) A set $\mathfrak{U}$ of rational quotients of $\mathscr{G}$ is *ample* if $\widehat{\Lambda}_\alpha \cap \widehat{\Lambda}_{\alpha'} = 0$ for any $\Phi_\alpha \neq \Phi_{\alpha'}$ in $\mathfrak{U}$, and if $\Lambda_{\mathfrak{U}} := \sum_{\Phi_\alpha \in \mathfrak{U}} \Lambda_\alpha$ is ℓ-adically dense in $\widehat{\Lambda}_{\mathscr{G}}$ and satisfies $\Lambda_{\mathfrak{U}} \cap \widehat{\mathfrak{U}}_{\mathscr{G}} = 0$.

(4) If $\mathfrak{U}$ is ample, then $\mathscr{G}$ is *geometric-like* with respect to $\mathfrak{U}$ if for every α, α' there exists a valuation v such that $\widehat{\Lambda}_\alpha, \widehat{\Lambda}_{\alpha'} \subseteq \widehat{U}_v$, and $\widehat{\Lambda}_\alpha$ and $\widehat{\Lambda}_{\alpha'}$ have the same (injective) image inside $\widehat{\Lambda}_{\mathscr{G}}$.

(5) Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-$\delta(= \delta_H)$ proper morphism of geometric-like (with respect to $\mathfrak{U}$ and $\mathfrak{B}$, respectively) abstract decomposition graphs. Then Φ is *compatible with rational quotients* if for each $\Psi_\beta \in \mathfrak{B}$, there exist $\Phi_\alpha \in \mathfrak{U}$ and $\Phi_{\alpha,\beta} : \mathscr{G}_\alpha \xrightarrow{\sim} \mathscr{H}_\beta$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{\Phi} & \mathscr{H} \\ \downarrow{\scriptstyle \Phi_\alpha} & & \downarrow{\scriptstyle \Psi_\beta} \\ \mathscr{G}_\alpha & \xrightarrow{\Phi_{\alpha,\beta}} & \mathscr{H}_\beta \end{array} \tag{5}$$

5.2. Remark. [3, Fact 32(2)] It can be shown that $\Lambda_\alpha := \hat{\phi}_\alpha(\Lambda_{\mathscr{G}_\alpha})$ can be group theoretically recovered from ($\widehat{\Lambda}_\alpha$ and) $\Lambda_{\mathscr{G}}$.

5.3. Proposition. *[3, Proposition 35] Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a level-$\delta(=\delta_H)$ proper morphism of geometric-like abstract decomposition graphs, which is compatible with the rational quotients $\mathfrak{U}$ and $\mathfrak{B}$. Then Φ is divisorial.*

5.4. Definition (Rational quotients – geometric approach)**.** For $t \in K$ non-constant, let $K_t = \overline{k(t)}^K$ be the relative algebraic closure of $k(t)$ in K. The prime divisors of K_t/k are in bijection with the closed points of its unique normal model $X_t \to k$, so $\mathscr{D}_K = \mathscr{D}_K^{\mathrm{tot}}$ is the unique maximal geometric prime divisor graph. The embedding $K_t \hookrightarrow K$ induces a projection $\Phi_{K_t} : \Pi_K^a \to \Pi_{K_t}^a$ and level-1 morphisms $\Phi_{K_t} : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\mathscr{D}_{K_t}}$ for any geometric decomposition graph $\mathscr{D}_K$ of K/k. Then Φ_{K_t} is a *rational quotient* of $\mathscr{D}_K$ in the sense of Definition 5.1 if and only if K_t is a rational function field, cf. [3, Proposition 41].

5.5. Definition (General elements and Bertini sets)**.**

(1) We call t a *general element* of K if $\overline{k(t)}^K = k(t)$.
(2) If $x, t \in K$ are algebraically independent over k and x is general (separable would suffice), then a "birational Bertini" argument shows that the elements $t_a = ax + t$ and $t_{a',a} = t/(a'x + a)$ and $t_{a'',a',a} = (a''t + a'x + a + 1)/(t + a'x + a)$ are general for almost all $a, a', a'' \in k$. These elements are *general elements of Bertini type.*
(3) A set $\Sigma \subset K^\times$ is a *Bertini set* if it contains all $t_a, t_{a',a}, t_{a'',a',a}$ for all $x, t \in K$ algebraically independent and x separable.
(4) Let $\mathfrak{U}_K = \{\Phi_{K_t}\}_{K_t}$ be the set of rational quotients of $\mathscr{G}_{\mathscr{D}_K}$ where $K_t = k(t)$ is generated by a general element t. A subset $\mathfrak{U} \subseteq \mathfrak{U}_K$ is of *Bertini type* if it contains $\mathfrak{U}_\Sigma = \{\Phi_{K_x} : x \in \Sigma\}$ for a Bertini set $\Sigma \subset K^\times$.

5.6. Proposition. *[1, Proposition 5.3(1)] Every abstract rational quotient (cf. Definition 5.1) is geometric (cf. Definition 5.4). That is, for every abstract rational quotient $\Phi_\alpha : \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}} \to \mathscr{G}_\alpha$ there exist a geometric rational quotient $\Phi_{K_x} : \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}} \to \mathscr{G}_{\mathscr{D}_{K_x}}$ for some general element $x \in K$, and an isomorphism $\Phi_{\alpha,K_x} : \mathscr{G}_\alpha \to \mathscr{G}_{\mathscr{D}_{K_x}}$ such that $\Phi_{K_x} = \Phi_{\alpha,K_x} \circ \Phi_\alpha$.*

5.7. Proposition.

(1) *The embedding $K_t \hookrightarrow K$ commutes with ℓ-adic completion.*
(2) *Suppose that there is an embedding of function fields $\iota : L/l \to K/k$. The diagram*

$$\begin{array}{ccc} \mathscr{G}_{\mathscr{D}_K} & \xrightarrow{\Phi_\iota} & \mathscr{G}_{\mathscr{D}_L} \\ \downarrow{\scriptstyle \Phi_{K_x}} & & \downarrow{\scriptstyle \Psi_{K_y}} \\ \mathscr{G}_{K_x} & \xrightarrow{\Phi_{K_x,K_y}} & \mathscr{G}_{K_y} \end{array} \tag{6}$$

(cf. Diagram (5)) commutes for any rational quotient K_x of K and all rational quotients K_y of L such that $\overline{\iota(K_y)}^K = \iota(K_y)$.
(3) *If $\iota(l) = k$ and $K/\iota(L)$ is separable, then general elements of Bertini type are mapped to general elements of Bertini type.*
(4) *Hence, in this case, if we have a proper morphism $\Phi_\iota : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}_{\mathscr{D}_L}$ of complete regular-like abstract decomposition graphs, then Φ_ι is compatible with some Bertini type sets $\mathfrak{U}$ and $\mathfrak{B}$ of rational quotients.*
(5) *A complete regular-like geometric decomposition graph $\mathscr{G}_{\mathscr{D}_K}$ endowed with a Bertini-type set $\mathfrak{U}$ of rational quotients is a geometric-like (cf. Definition 5.1(4)) abstract decomposition graph.*

6. Global theory

6.1. Idea. We can (Galois) group theoretically characterise geometric decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ inside $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$.

- A connected full subgraph $\mathscr{G}_{\mathscr{D}} \subset \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ is a geometric decomposition graph if and only if every vertex Kv of $\mathscr{G}_{\mathscr{D}}$ corresponds to a geometric set of prime divisors D_v and all maximal oriented branches of $\mathscr{G}_{\mathscr{D}}$ have length $\mathrm{td}(K/k)$.
- When $\mathrm{td}(Kv/k) = 1$, a non-empty set D_v of prime divisors of Kv/k is geometric if and only if Π^a_{Kv}/T_{D_v} is topologically finitely generated, where $T_{D_v} = \langle T_w \text{ s.t. } w \in D_v \rangle$. For higher transcendence degrees, similar conditions are obtained by induction.
- Hence, geometric sets of prime divisors of Kv/k can be recovered from $\mathscr{G}_{\mathscr{D}_{K_v^{\mathrm{tot}}}}$.
- We can group theoretically recover $\mathscr{G}_{\mathscr{D}_{K_v}^{\mathrm{tot}}}$ from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ as the subgraph corresponding to valuations $w \geq v$.
- Hence, geometric sets of prime divisors of Kv/k can be recovered from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$. These determine geometric decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$.

6.2. Definition (Complete regular-like)**.** A geometric set D_X of prime divisors for K/k is *complete regular-like* if three technical conditions hold on the completed class group $\widehat{\mathfrak{Cl}}(D)$. For instance, D_X is complete regular-like if $X \to k$ is a complete regular variety (but the converse is not true in general).
A level-δ geometric prime divisor graph $\mathscr{D}_K$ and its geometric decomposition graph $\mathscr{G}_{\mathscr{D}_K}$ are said to be *complete regular-like* if for every vertex v such that $\mathrm{td}(Kv/k) > 0$, the set D_v of non-trivial valuations of Kv inside $\mathscr{D}$ is complete regular-like.
Every geometric prime divisor graph is contained in a complete regular-like one.

6.3. Proposition. *[3, Proposition 22] The complete regular-like decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ can be group theoretically recovered from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$.*

Sketch of the proof. As mentioned in Idea 6.1, for any vertex Kv of $\mathscr{D}_K$, we can group theoretically recover $\mathscr{G}_{\mathscr{D}_{K_v}^{\mathrm{tot}}}$ from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$. In particular, we recover the inertia groups T_w for prime divisors w of Kv/k and the closed subgroup T_{Kv} they generate. These data allow us to check (by comparing T_{Kv} to T_{D_v}, among other things) whether $\mathscr{G}_{\mathscr{D}_K}$ is complete regular-like. □

6.4. Lemma. *If a geometric decomposition graph $\mathscr{G}_{\mathscr{D}_K}$ is complete regular-like, then as a level-δ pro-ℓ abstract decomposition graph (cf. Proposition 4.2(3)) it is divisorial (cf. Definition 4.3).*

6.5. Lemma. *[3, Proposition 11] If $\mathscr{G}$ is a level-δ complete curve-like (cf. Definition 4.1(7)) abstract decomposition graph and satisfies another technical condition (of being* ample up to level δ*), then any two abstract divisor groups $\mathrm{Div}_{\mathscr{G}}$, $\mathrm{Div}'_{\mathscr{G}}$ are ℓ-adically equivalent (i.e., there exists $\epsilon \in \mathbb{Z}_\ell^\times$ such that $\mathrm{Div}_{\mathscr{G}} = \epsilon \cdot \mathrm{Div}'_{\mathscr{G}}$) inside $\widehat{\mathrm{Div}}_{\mathscr{G}}$. (Then the corresponding inertia generator systems $\mathfrak{I}$, $\mathfrak{I}'$ are also ℓ-adically equivalent.)*

6.6. Lemma. *[3, Remarks 19] Let $\mathscr{G}_{\mathscr{D}_K}$ be a level-δ pro-ℓ abstract decomposition graph with origin $G = \Pi^a_K$. Let $\widehat{\Lambda}_{\mathscr{G}_{\mathscr{D}_K}} := \mathrm{Hom}(\Pi^a_K, \mathbb{Z}_\ell)$ and recall Definition 4.3. Choose a normal model $X \to k$ of K/k such that D_X is the set of* 1*-residual edges of $\mathscr{D}_K$; then this comes with an exact sequence*

$$0 \to \widehat{U}_D \to \widehat{K} \to \widehat{\mathrm{Div}}_D \to \widehat{\mathfrak{Cl}}(D) \to 0 \tag{7}$$

which we can identify term-by-term with the sequence in (2)*, to give a geometric interpretation for each of the objects appearing there. In particular, by Kummer theory,* $\widehat{\Lambda}_{\mathscr{G}_{\mathscr{D}_K}} = \widehat{K}$.

6.7. Idea. To recover $K^\times/k^\times$ from $\widehat{K}$, it will be key to construct a suitable arithmetic divisorial lattice $\oplus_v \mathbb{Z}_{(\ell)} v$ (i.e., an abstract divisor group, cf. Definition 4.3) which fits into a short exact sequence

$$0 \to (K^\times/k^\times)_{(\ell)} \to \oplus_v \mathbb{Z}_{(\ell)} v \to (\mathrm{Pic})_{(\ell)} \to 0.$$

Let Λ be the preimage of $\oplus_v \mathbb{Z}_{(\ell)} v$ under $(\widehat{\Lambda}_{\mathscr{G}} \simeq) \widehat{K} \to \widehat{\mathrm{Div}}_{\mathscr{G}}$. Then Λ will contain $K^\times/k^\times$, and the cokernel of $K^\times/k^\times \hookrightarrow \Lambda$ should be ℓ^∞-torsion.

6.8. Idea (An embedding of function fields induces a morphism of abstract decomposition graphs)**.** [3, Proposition 38]

- Let k and l be two algebraically closed fields of characteristic $\neq \ell$ and let $\iota : L/l \hookrightarrow K/k$ be an embedding of function fields such that $\iota(l) \simeq k$ and $K/\iota(L)$ is separable. Then ι induces a surjective morphism $\phi_\iota : \mathscr{D}_K^{\mathrm{tot}} \to \mathscr{D}_L^{\mathrm{tot}}$.
- A prolongation $\iota' : L' \to K'$ induces a canonical surjective map $\Phi_\iota : \Pi_K^a \to \Pi_L^a$. For v, w valuations of K, L respectively, inertia generators τ_v, τ_w are related via $\Phi_\iota(\tau_v) = \tau_w^{[vK:wL]}$.
- If $K/\iota(L)$ is finite (hence algebraic), then ϕ_ι also maps geometric decomposition graphs surjectively to geometric decomposition graphs. Otherwise (replacing $\iota(L)$ by $\overline{\iota(L)}^K$ if necessary), $K/\iota(L)$ is regular. By carefully choosing our models for K and L, we can prove that for geometric decomposition graphs $\mathscr{D}_K$ and $\mathscr{D}_L$ there exists a unique maximal geometric subgraph $\mathscr{D}'_K \subset \mathscr{D}_K$ such that there is a morphism $\phi_\iota : \mathscr{D}'_K \to \mathscr{D}_L$. And for some geometric decomposition graphs $\mathscr{D}_K^0 \supseteq \mathscr{D}_K$ and $\mathscr{D}_L^0 \supseteq \mathscr{D}_L$, the map $\phi_\iota : \mathscr{D}_K^0 \to \mathscr{D}_L^0$ is surjective.
- Hence, ι induces a level-$\mathrm{td}(L/l)$ morphism $\Phi_\iota : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\mathscr{D}_L}$ of abstract decomposition graphs.
- If ϕ_ι is proper, then Φ_ι is proper. And if Φ_ι is proper and $\mathscr{G}_{\mathscr{D}_K}, \mathscr{G}_{\mathscr{D}_L}$ are complete regular-like, then Φ_ι is divisorial (cf. Lemma 6.4).
- The map ι defines Φ_ι uniquely, while Φ_ι determines ι up to Frobenius twist. This follows from the fact that $\hat{\phi} : \widehat{L} \to \widehat{K}$ is the ℓ-adic completion of ι, and from Lemma 6.5.

With this idea in mind, we start proving the converse by deducing the following from our previous results:

6.9. Proposition. *[3, Proposition 39] Let* $\Phi : \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}} \to \mathscr{H}_{\mathscr{D}_L^{\mathrm{tot}}}$ *be an isomorphism of total decomposition graphs.*

(1) *For each geometric decomposition graph* $\mathscr{G}_{\mathscr{D}_K}$ *for* K/k*, there exists a geometric decomposition graph* $\mathscr{G}_{\mathscr{D}_L}$ *for* L/l *such that* $\Phi : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{G}_{\mathscr{D}_L}$ *is an isomorphism.*
(2) *The graph* $\mathscr{G}_{\mathscr{D}_K}$ *is regular-like if and only if* $\mathscr{G}_{\mathscr{D}_L}$ *is regular-like. If this is the case, then* Φ *is divisorial.*

Sketch of proof. By Proposition 6.3, the (complete regular-like) decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ and $\mathscr{H}_{\mathscr{D}_L}$ can be group theoretically recovered from $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ and $\mathscr{H}_{\mathscr{D}_L^{\mathrm{tot}}}$ respectively, in such a way that Φ restricted to $\mathscr{G}_{\mathscr{D}_K}$ is an isomorphism. By Lemma 6.4, the graphs $\mathscr{G}_{\mathscr{D}_K}$ and $\mathscr{H}_{\mathscr{D}_L}$ are divisorial as abstract decomposition graphs. Using Lemma 4.6, one shows that Φ is divisorial. □

6.10. Remark. Proposition 6.9 in particular implies that to recover (an isomorphism between) function fields from (an isomorphism between) total decomposition graphs, it suffices to start with (a divisorial isomorphism between) complete regular-like geometric decomposition graphs. Viewing these (complete regular-like) geometric decomposition graphs as (divisorial) abstract decomposition graphs, Proposition 5.7(4) implies that such an isomorphism is always compatible with some Bertini type sets of rational quotients.

6.11. Theorem (Main theorem of global theory)**.**
Let K/k and L/l be function fields of transcendence degrees > 1, with complete regular-like geometric decomposition graphs $\mathscr{G}_{\mathscr{D}_K}$ and $\mathscr{H}_{\mathscr{D}_L}$ (viewed as geometric-like abstract decomposition graphs) endowed with Bertini-type sets $\mathfrak{U}$ and $\mathfrak{V}$ of rational quotients, respectively. Let $\Phi : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}_{\mathscr{D}_L}$ be a proper morphism compatible with $\mathfrak{V}$ and $\mathfrak{U}$, induced from an open group homomorphism $\Phi : \Pi_K^a \to \Pi_L^a$.
Then there exists an ℓ-adic unit ϵ, and an embedding $\iota : L/l \hookrightarrow K/k$ of function fields which induces a morphism $\Phi_\iota : \mathscr{G}_{\mathscr{D}_K} \to \mathscr{H}_{\mathscr{D}_L}$ as in Idea 6.8, such that $\Phi = \epsilon \cdot \Phi_\iota$.
If $\iota(l) = k$ then ι is unique up to Frobenius twists.

Outline of proof. Let $j_K : K^\times \to \widehat{K}$ resp. $j_L : L^\times \to \widehat{L}$ denote the ℓ-adic completion functors, and define $K_{(\ell)}^\times := j_K(K^\times) \otimes \mathbb{Z}_{(\ell)}$ and $L_{(\ell)}^\times := j_L(L^\times) \otimes \mathbb{Z}_{(\ell)}$. Without loss of generality, we may assume that the Kummer morphism $\hat{\phi} : \widehat{\Lambda}_{\mathscr{H}_{\mathscr{D}_L}} = \widehat{L} \to \widehat{\Lambda}_{\mathscr{G}_{\mathscr{D}_K}} = \widehat{K}$ maps $L_{(\ell)}^\times$ isomorphically to $K_{(\ell)}^\times$.
For $\Phi_{K_x} \in \mathfrak{U}$ corresponding to $\Phi_{K_y} \in \mathfrak{V}$, let $K_{x,(\ell)} \subset \widehat{K_x}$ and $K_{y,(\ell)} \subset \widehat{K_y}$ be the unique respective divisorial lattices such that $\hat{\phi}_{K_x}(K_{x,(\ell)}) \subset K_{(\ell)}^\times$ and $\hat{\phi}_{K_y}(K_{y,(\ell)}) \subset L_{(\ell)}^\times$. Then it follows (from Proposition 5.7, among other things) that $\hat{\phi}(K_{y,(\ell)}) = K_{x,(\ell)}$. Without loss of generality, we may assume that $\hat{\phi} \circ j_L(y) = j_K(x)$, in which case $\hat{\phi}$ induces a bijection $j_L(K_y^\times) \to j_L(K_x^\times)$.
Let $M_K := \hat{\phi}(j_L(L^\times)) \cap j_K(K^\times)$ and $M_L := \hat{\phi}^{-1}(M_K) \subseteq j_L(L^\times)$. Then $\hat{\phi} : M_L \to M_K$ is an isomorphism mapping $j_L(K_y^\times)$ isomorphically to $j_K(K_x^\times)$.
Now let $K_0 = j_K^{-1}(M_K) \cup \{0\} \subseteq K$ and $L_0 = j_L^{-1}(M_L) \cup \{0\}$. Then these are function subfields. (One needs to check they are closed under addition, cf. [3, Lemmas 49 and 50].)
The connection to projective geometry follows from the fact that $M_K = j_K(K_0^\times) = K_0^\times/k^\times = \mathbb{P}(K_0)$ and $M_L = \mathbb{P}(L_0)$. These projective spaces are mapped bijectively to each other under $\phi := \hat{\phi}|_{\mathbb{P}(L_0)}$.
A line in $\mathbb{P}(K_0)$ is $\mathfrak{l}_{t_0,t_1} = (kt_0 + kt_1)^\times/k^\times = t_0 \cdot ((k(t_1/t_0) + k)^\times/k^\times) = t_0 \cdot j_K(kt + k)^\times$ for some t_0 and t_1 linearly independent and $t = t_1/t_0$; thus, the line only depends on $j_K(t_0)$ and $j_K(t_1)$. Similarly one considers lines in $\mathbb{P}(L_0)$.
The map ϕ maps lines in $\mathbb{P}(L_0)$ bijectively to lines in $\mathbb{P}(K_0)$, cf. [3, Lemma 51].
Now the statement follows from applying Artin's fundamental theorem of projective geometries, after applying a suitable scaling. The existence of ϵ follows from Lemma 6.5. Uniqueness up to Frobenius twist follows from the fact that L/L_0 is purely inseparable, cf. [3, Lemma 52]; to prove this, one uses the observation that $M_L \otimes \mathbb{Z}_{(\ell)} = L_{(\ell)}^\times \subset \widehat{L}$. $\square$

7. Local theory

7.1. Theorem. *[2, Theorem 4.4]*
Let $pr : \Pi_K^c \to \Pi_K^a$ be the projection map and for any subgroup $G \subseteq \Pi_K^a$, let $G := pr^{-1}(G)$.

(1) *The transcendence degree $\mathrm{td}(K/k)$ can be recovered from Π_K^a as the maximal integer d for which there exists a closed subgroup $\Delta \subseteq \Pi_K^a$ such that $\Delta \simeq \mathbb{Z}_\ell^d$ and Δ is commuting-liftable.*

(2) *Choose an integer $r < d := \mathrm{td}(K/k)$ and $T \subseteq Z$ closed subgroups of Π_K^a. Then $T \subseteq Z$ is an r-divisorial subgroup of Π_K^a if and only if Z is maximal among the closed subgroups of Π_K^a such that:*
   (a) *the group Z contains a closed subgroup $\Delta \simeq \mathbb{Z}_\ell^d$ such that Δ is commuting-liftable;*
   (b) *the group $T \simeq \mathbb{Z}_\ell^r$ and $T' = Z(Z')$.*

Sketch of proof.

(1) If Δ is a commuting-liftable *non*-pro-cyclic subgroup of Π_K^a, then by Corollary 3.6(1), there exists a valuation v such that $\Delta \subseteq Z_v$ and $\Delta/(\Delta \cap T_v)$ is pro-cyclic. Now the proof follows from Notation 2.3 and Abhyankar's inequality $\mathrm{rank}(vK) + \mathrm{td}(Kv) \leq \mathrm{td}(K/k)$.

(2) One direction follows from Remark 3.2(3) and Lemma 3.3. We prove the converse.
   Again by Corollary 3.6(1), there exists a valuation v_0 such that $Z \subseteq Z_{v_0}$ and $T = Z \cap T_{v_0}$. Take $\Lambda = (K')^T$ and let v be the abelian pro-ℓ Λ-core of v_0.
   Let Δ be a maximal commuting-liftable subgroup of Z such that $\Delta \simeq \mathbb{Z}_\ell^d$. (Note that necessarily $T \subseteq \Delta$, by maximality of Δ.) Applying Corollary 3.6(1) to Δ yields a valuation v_Δ such that $\Delta \subseteq Z_{v_\Delta}$; let w be its abelian pro-ℓ Λ-core.
   By comparing (the maximal pro-ℓ extensions of) their residue fields, it follows that $v = w$. By construction, $T \subseteq T_v$ and $Z \subseteq Z_v$, so by maximality of T and Z, these groups are equal, as required.

□

7.2. Corollary.

(1) *The r-divisorial subgroups can be group theoretically recovered from Π_K^a (hence from Π_K^c). In fact, an isomorphism $\Pi_K^a \to \Pi_L^a$ induced from an isomorphism $\Pi_K^c \to \Pi_L^c$ bijectively maps (flags of) r-divisorial subgroups to (flags of) r-divisorial subgroups.*

(2) *A sequence $Z_1 \geq \ldots \geq Z_r$ of closed subgroups of Π_K^a with a corresponding sequence $T_{v_1} \leq \ldots \leq T_{v_r}$ of closed subgroups $T_i \subseteq Z_i$ is a flag of r-divisorial subgroups if and only if the groups are maximal such that the following hold:*
   (a) *each Z_i contains $\Delta \simeq \mathbb{Z}_\ell^d$ such that Δ' is abelian;*
   (b) *each T_i satisfies $T_i \subset \mathfrak{In}(K)$ and $T_i \simeq \mathbb{Z}_\ell^i$ and $T_i' = Z(Z_i')$.*

(3) *For any r-divisor v of K/k, the r-divisorial subgroups of Π_{Kv}^a are exactly the images under $Z_v \to \Pi_{Kv}^a$ (cf. (1)) of the r-divisorial subgroups $Z \subseteq Z_v$ endowed with $T \supseteq T_v$.*

(4) *Hence, the total decomposition graph $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$ can be group theoretically recovered from Π_K^c.*

7.3. Proposition. *[1, Proposition 5.3(2)] Any isomorphism of total decomposition graphs $\Phi : \mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}} \to \mathscr{H}_{\mathscr{D}_L^{\mathrm{tot}}}$ is compatible with (geometric) rational quotients. That is, for each geometric rational quotient $\Phi_{L_y} : \mathscr{H}_{\mathscr{D}_L^{\mathrm{tot}}} \to \mathscr{G}_{\mathscr{D}_{L_y}}$, the morphism $\Phi_\alpha := \Phi_{L_y} \circ \Phi$ determines a(n abstract, hence by Proposition 5.6 a) geometric rational quotient of $\mathscr{G}_{\mathscr{D}_K^{\mathrm{tot}}}$.*

Sketch of proof. It follows from Proposition 6.9(1) and Idea 6.1 that for every $v \in \mathscr{D}_K$ which is mapped to $w \in \mathscr{D}_L$, the residual morphism $\Phi_v : \mathscr{G}_{\mathscr{D}_{Kv}} \to \mathscr{H}_{\mathscr{D}_{Lw}}$ is an isomorphism. By Lemma 4.6, each Φ_v is divisorial. So after dualising, each $\hat{\phi}_v : \widehat{Lw^\times} \to \widehat{Kv^\times}$ is an isomorphism, and there is a commutative diagram

$$\begin{array}{ccc} \widehat{U_{\mathscr{G}_{\mathscr{D}_{Lw}}}} & \xrightarrow{\hat{\phi}} & \widehat{U_{\mathscr{G}_{\mathscr{D}_{Kv}}}} \\ \downarrow & & \downarrow \\ \widehat{Lw^\times} & \xrightarrow{\hat{\phi}_v} & \widehat{Kv^\times} \end{array} . \tag{8}$$

If v and w are 1-vertices, then we also obtain a commutative diagram

$$\begin{array}{ccc} \widehat{L^\times} & \longrightarrow & \mathbb{Z}_\ell \\ \downarrow{\scriptstyle \hat{\phi}} & & \downarrow{\scriptstyle a_{vw}} \\ \widehat{K^\times} & \longrightarrow & \mathbb{Z}_\ell \end{array} \tag{9}$$

for some $a_{vw} \in \mathbb{Z}_{(\ell)}$. Using these diagrams, one checks that $\Phi_\alpha := \Phi_{L_y} \circ \Phi$ is surjective, divisorial, and determines a rational quotient, cf. Definition 5.1. □

7.4. Remark.

(1) Since we are assuming that k is the algebraic closure of a finite field, it only carries the trivial valuation. In particular, the *quasi prime r-divisors* of [1] are just prime r-divisors as in Definition 2.2.

(2) It is possible to determine from Π_K^a whether k is the algebraic closure of a finite field, since this is equivalent to the fact that there are generators τ_w for all r-divisorial inertia groups T_w such that $\prod_w \tau_w = 1$ is the only pro-relation on the system $(\tau_w)_w$, cf. [1, Lemma 4.2].

References

[1] Florian Pop, *On the birational anabelian program initiated by Bogomolov I*, Inventiones Mathematicae **(187)** no. 3, 2012, pp. 511–533.

[2] Florian Pop, *Pro-ℓ abelian-by-central Galois theory of prime divisors*, Israel Journal of Mathematics **(180)**, 2010, pp. 43–68.

[3] Florian Pop, *Recovering function fields from their decomposition graphs*, Number theory, analysis and geometry, Springer New York, 2012, pp. 519–594.

[4] Florian Pop, *Reconstruction of Function Fields from their pro-ℓ Abelian divisorial inertia*, Preprint available at `https://www.math.upenn.edu/~pop/Research/files-Res/2017Oct16_BP-DI.pdf`, 2016.

[5] Florian Pop, *Inertia elements versus Frobenius elements*, Mathematische Annalen **(438)**, 2010, pp. 1005–1017.

[6] Fedor Bogomolov and Yuri Tschinkel, *Commuting elements of Galois groups of function fields*, Motives, polylogarithms and Hodge theory, Part I, International Press Lecture Series **3**, International Press Somerville MA, 2002, pp. 75–120.

E-mail address: `vkarem@math.upenn.edu`